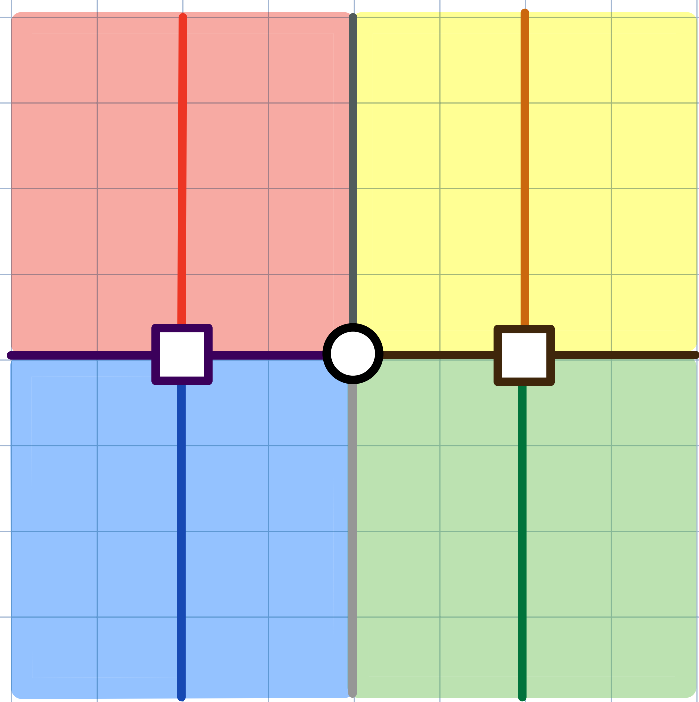
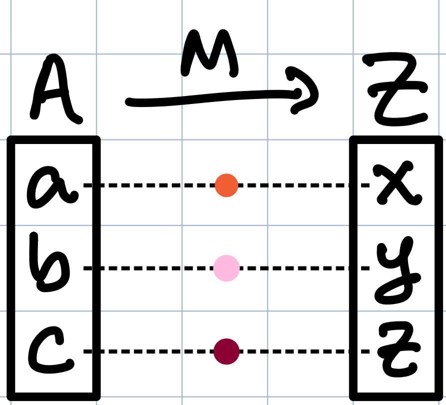
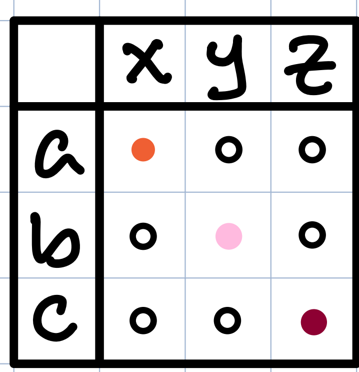
logic in color

monads & modules



We saw a relation as a 0/1-matrix, and realized: we can have many kinds of data — judgements generalize relations.

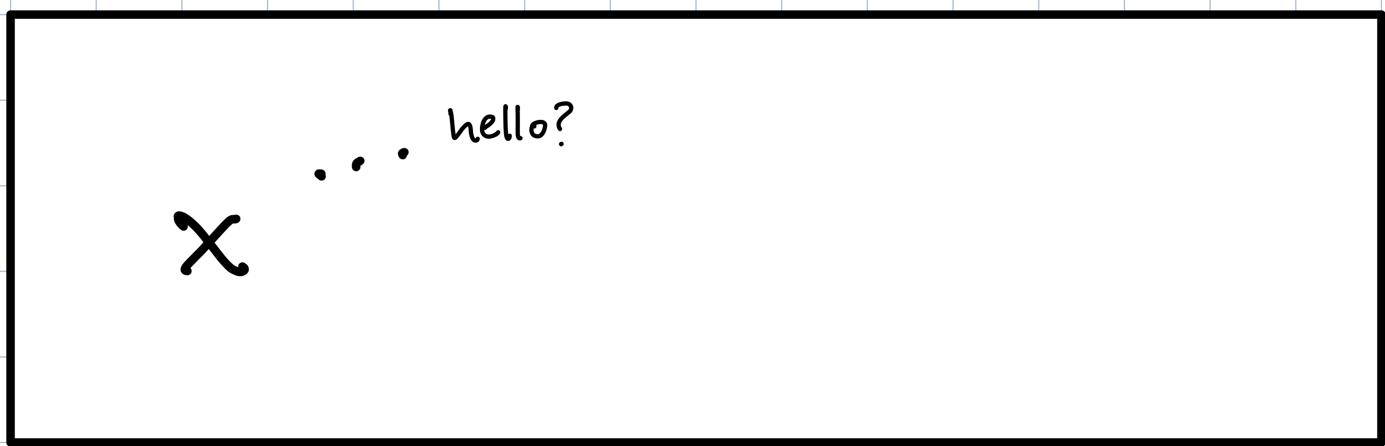




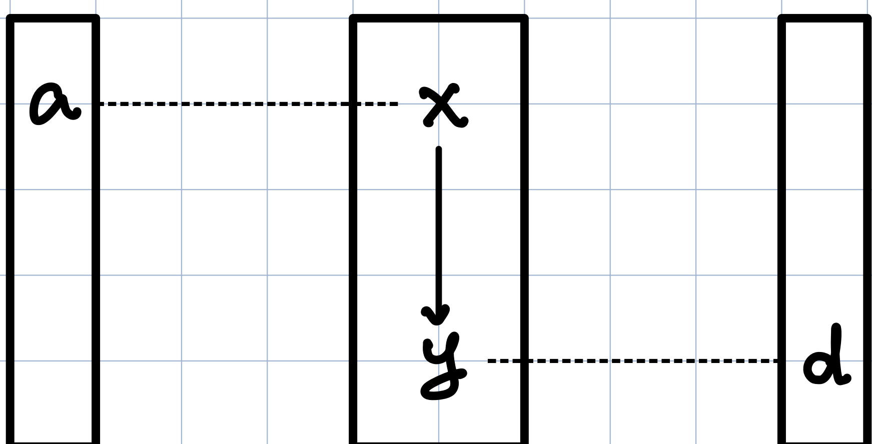
Only one problem: sets don't "understand"!

In the "logic of distance", for example, types need distance too.

Otherwise, in a set, everything is infinitely far away from each other!



* Internalizing the notion of judgement allows logic to flow through types.



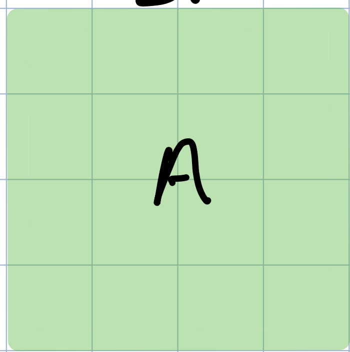
In other words, "as above, so below".

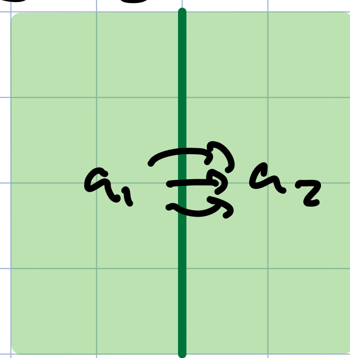
Let $\mathbb{C}$ be a dbl cat. (with inf-quotients)

Monad

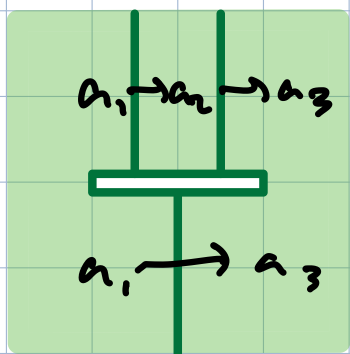
($\mathbb{C} = \mathrm{Mat}\,\mathbb{V}$)

A type & judgement

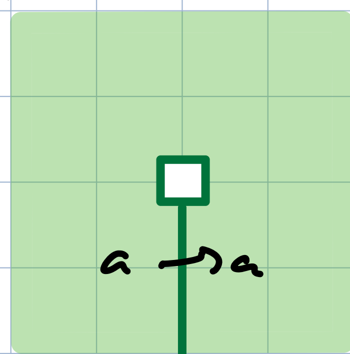




with inferences



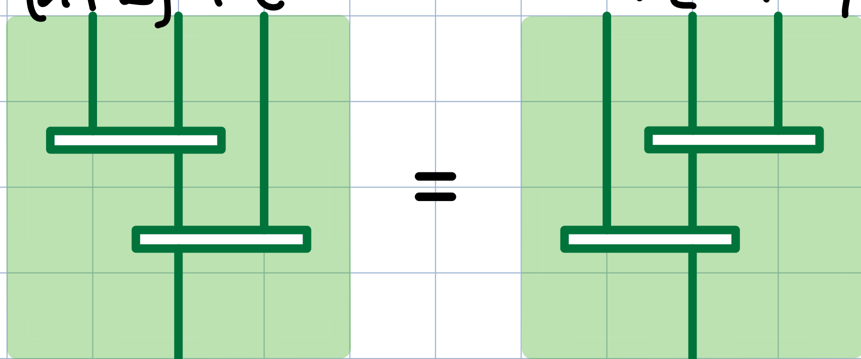
"join"



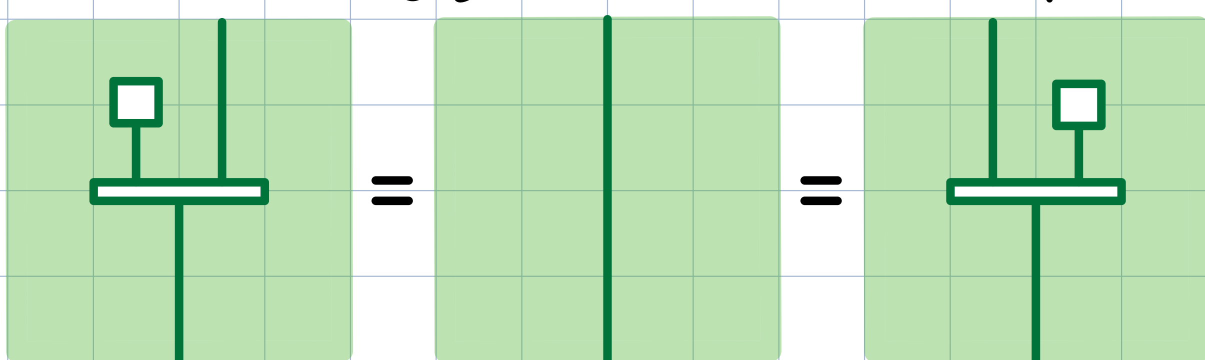
"unit"

which are

$(a+b)+c$ $a+(b+c)$



$0+a$ associative $a+0$

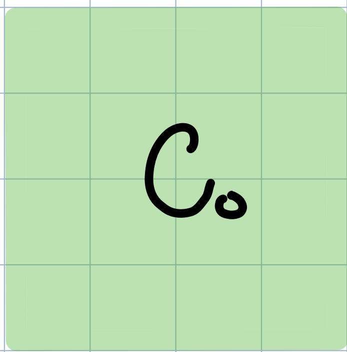


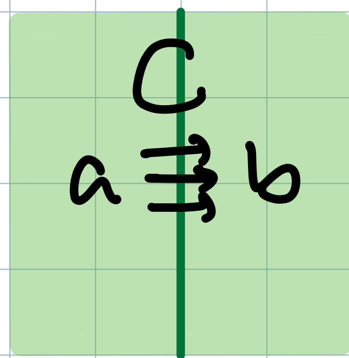
& left & right unital.

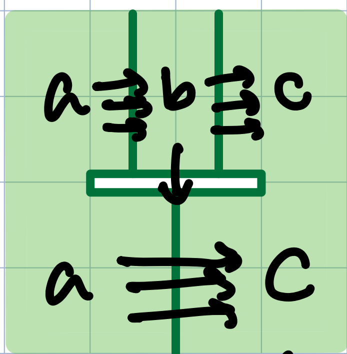
[note: identity is a monad.]

ex: matrices of $(\mathrm{Set}, \times, \Sigma)$

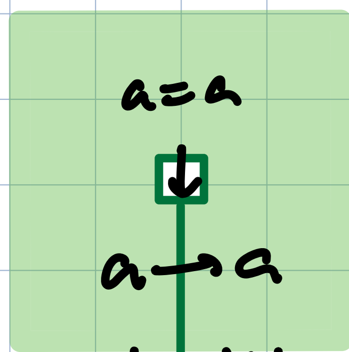
aka "Span"



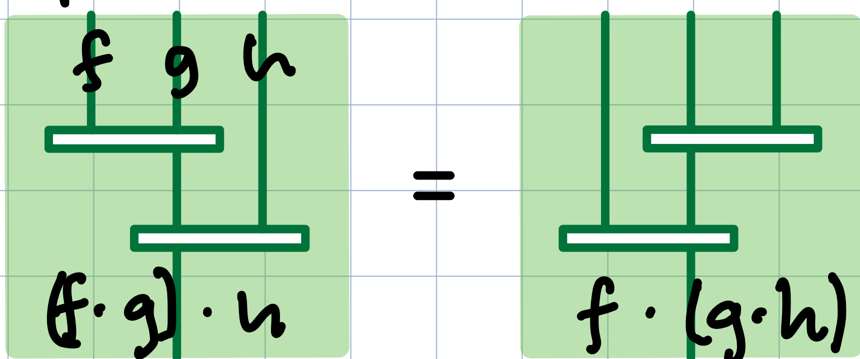


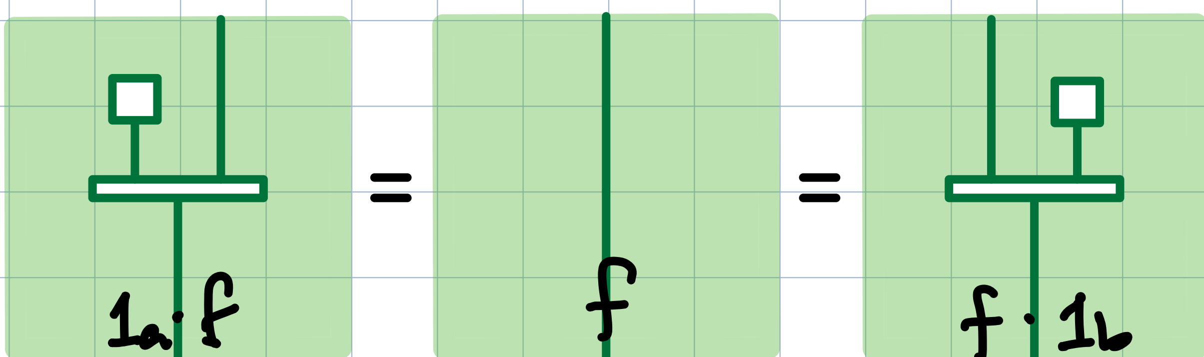


composition



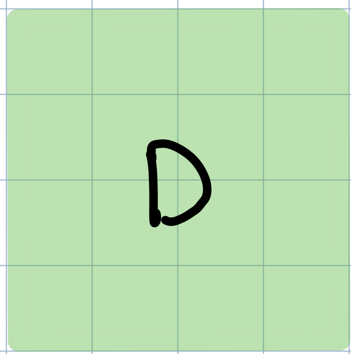
identity

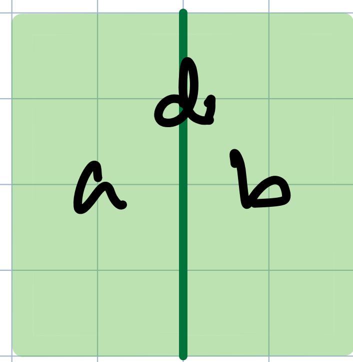


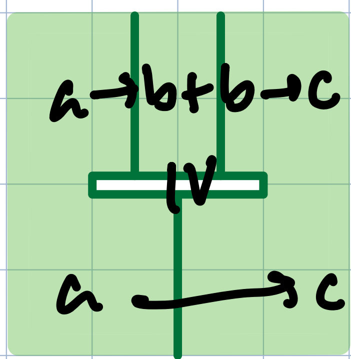


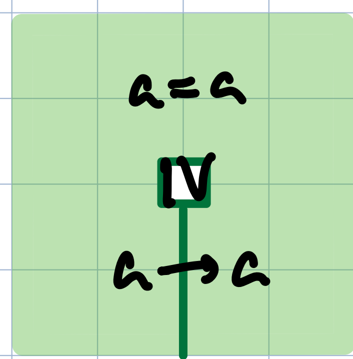
category!

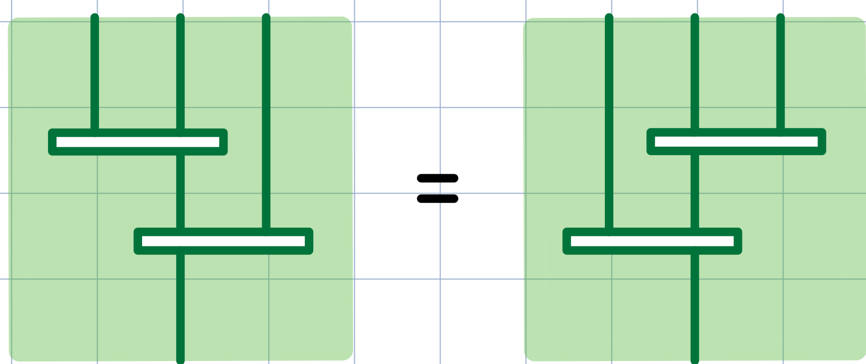
ex: matrices of $((\overline{\mathbb{R}}^+, \geq), +, \inf)$

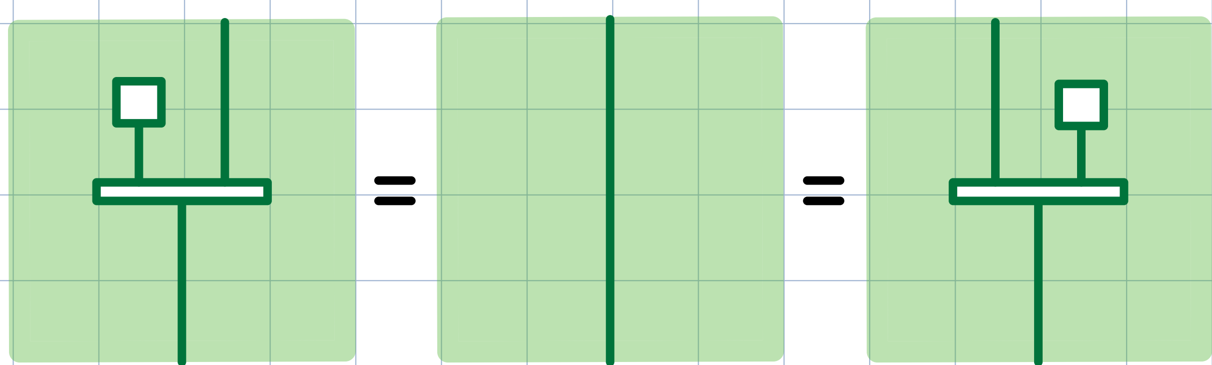






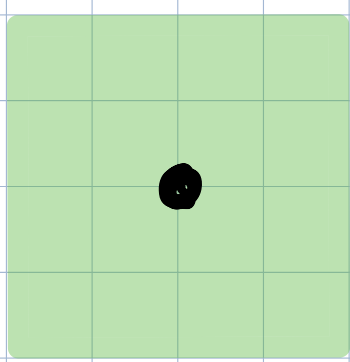


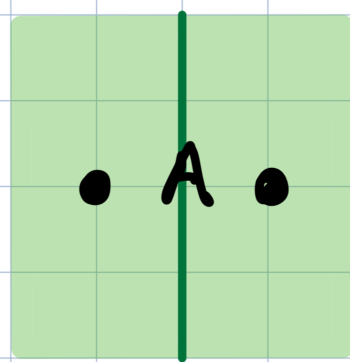


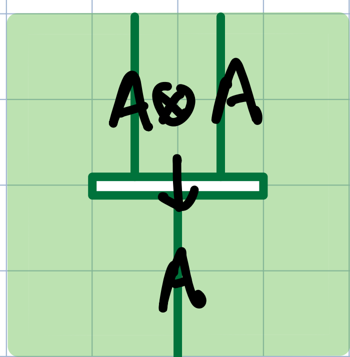


(Lawvere) metric space!

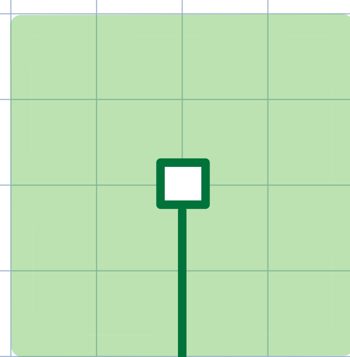
ex: matrices of $(\mathrm{Ab}, \otimes, \Sigma)$



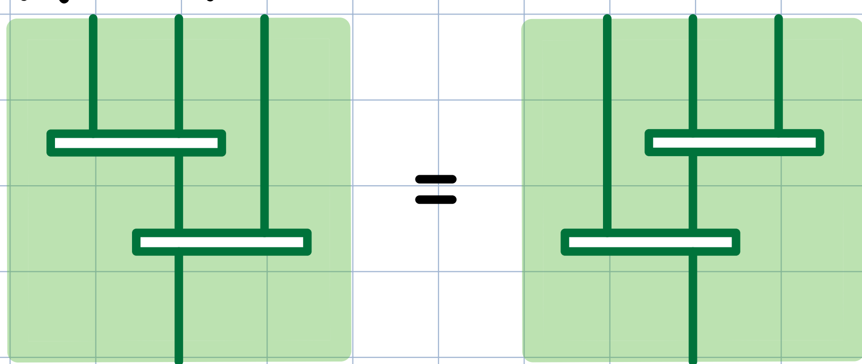


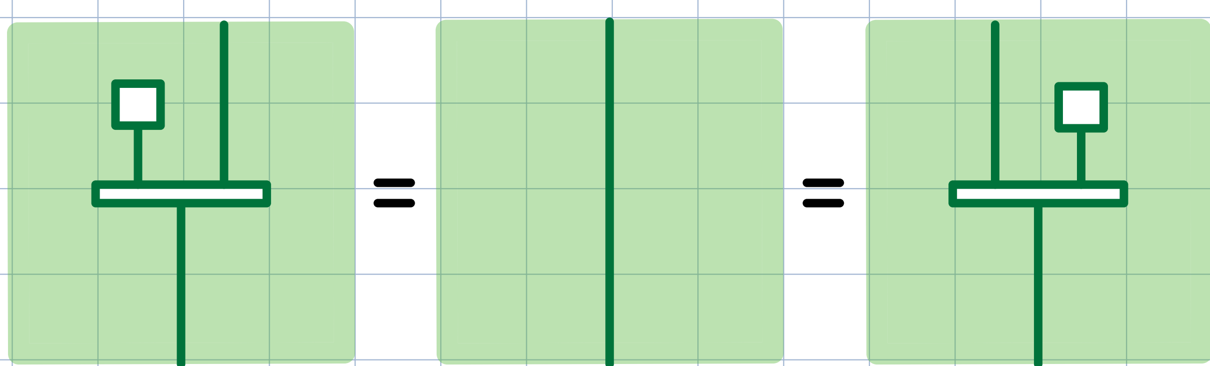


multiplication



identity



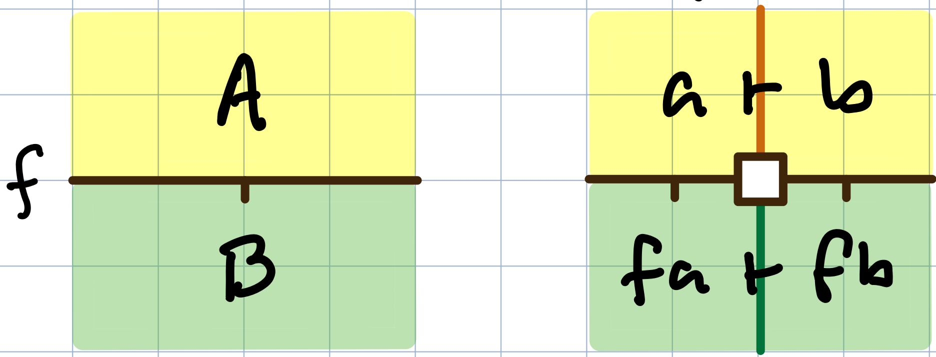


ring!

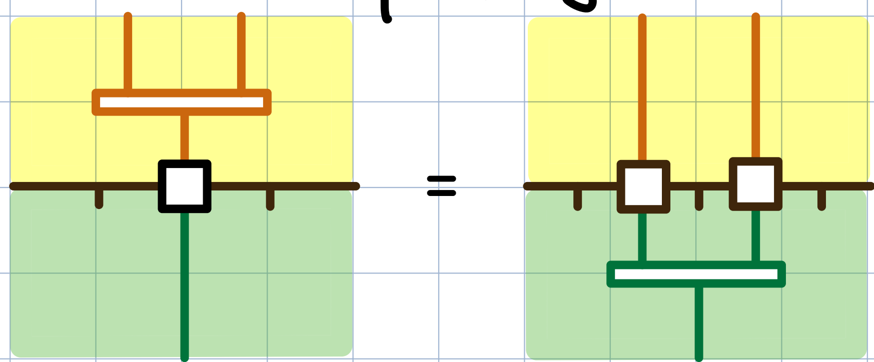
If monads are types, what are terms?

Morphism:

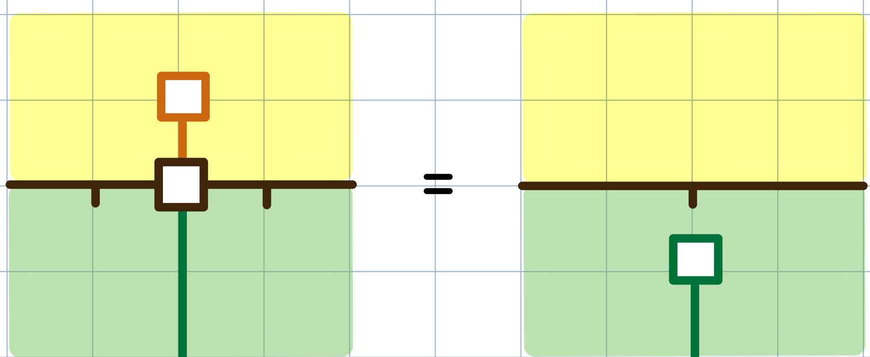
a term + inference



respecting



join



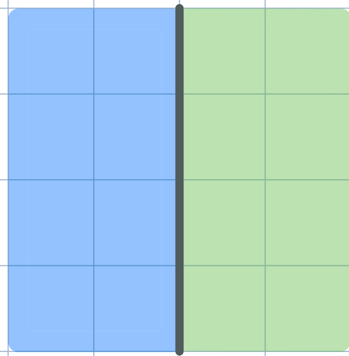
+ unit.

exs:

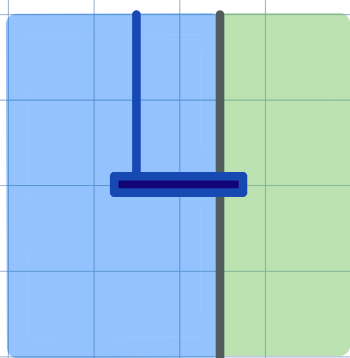
$\mathbb{B}$	monotone map
Set	functor
$\mathbb{R}$	short map
Ab	homomorphism

What is a judgement between monads?

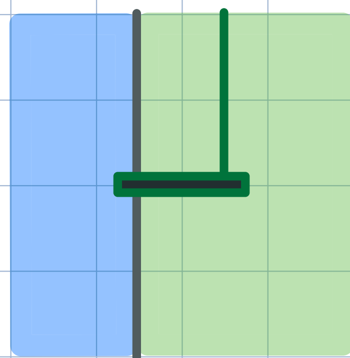
biModule:
a judgement



with

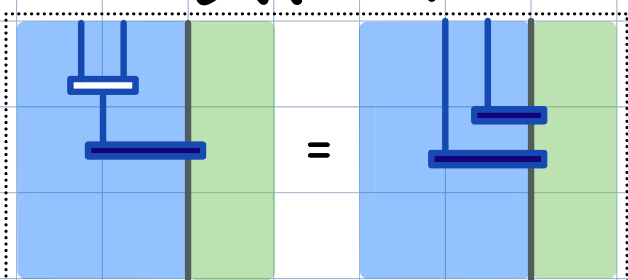


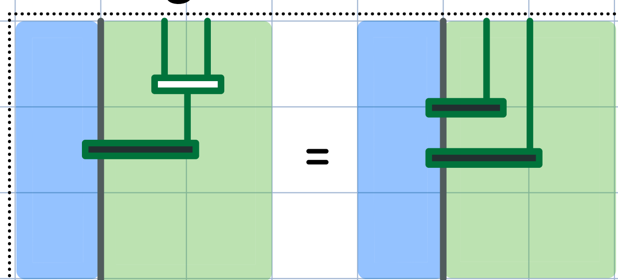
inferences



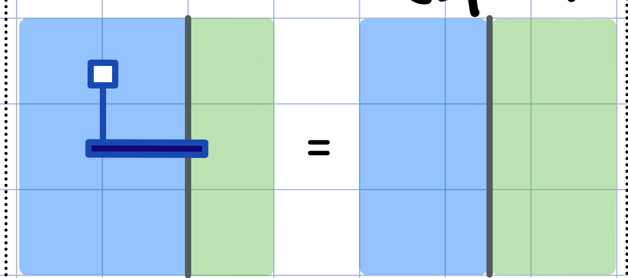
left action

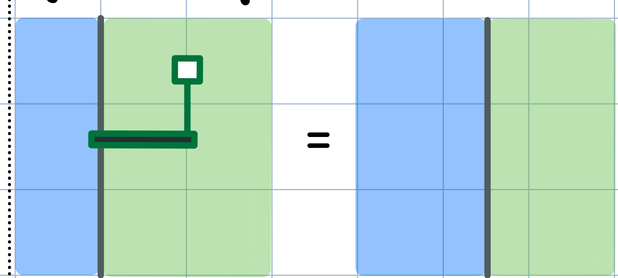
right action



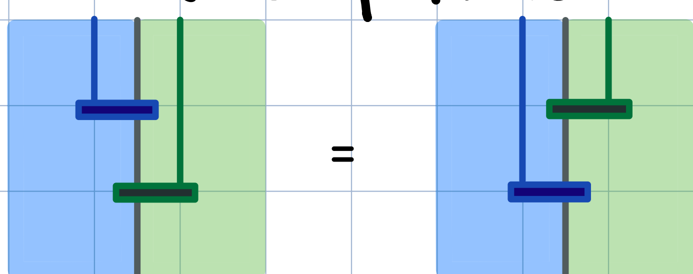


respecting join + unit



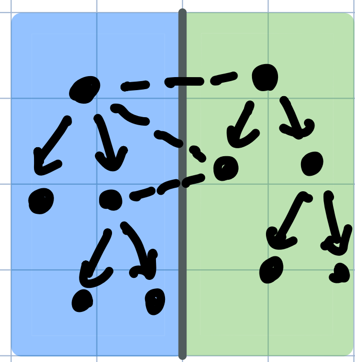


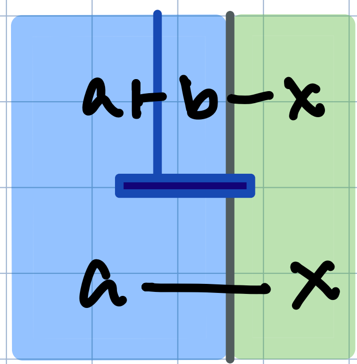
and compatible.

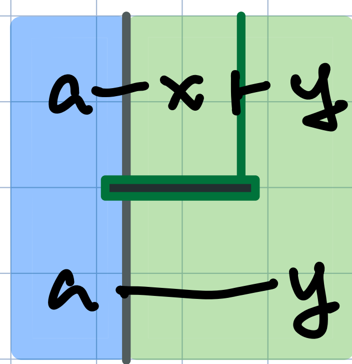


* Judgements are interactive! *

ex: B

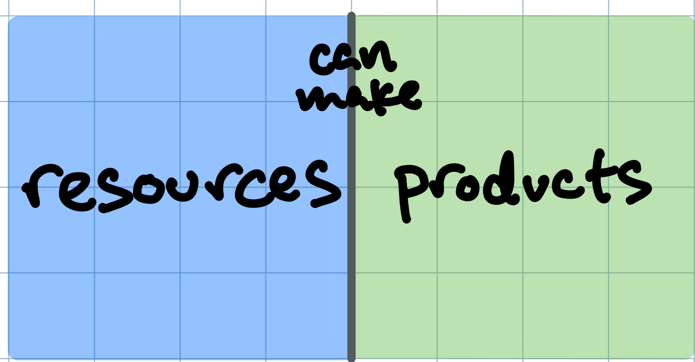


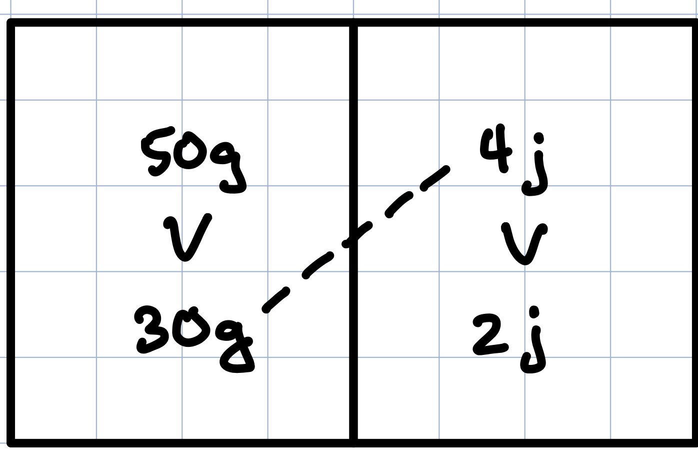




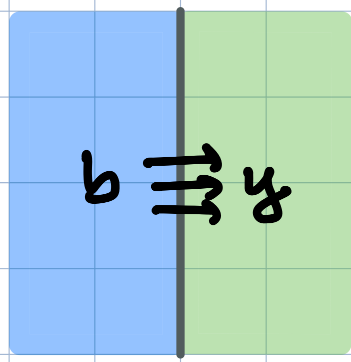
(equations are automatic)

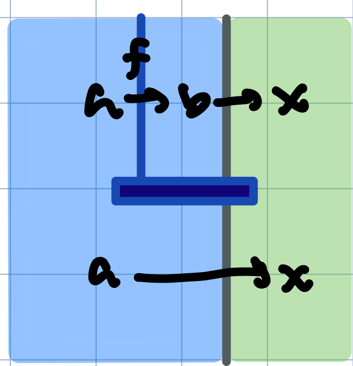
Intuition

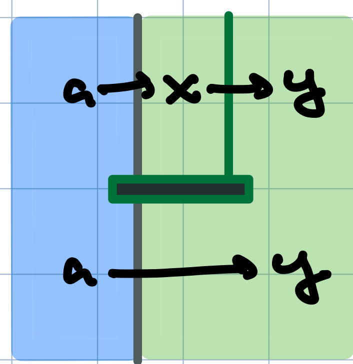


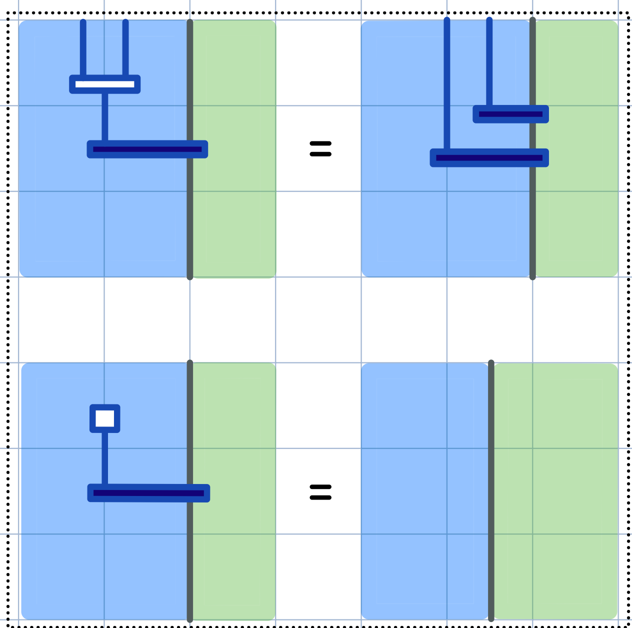


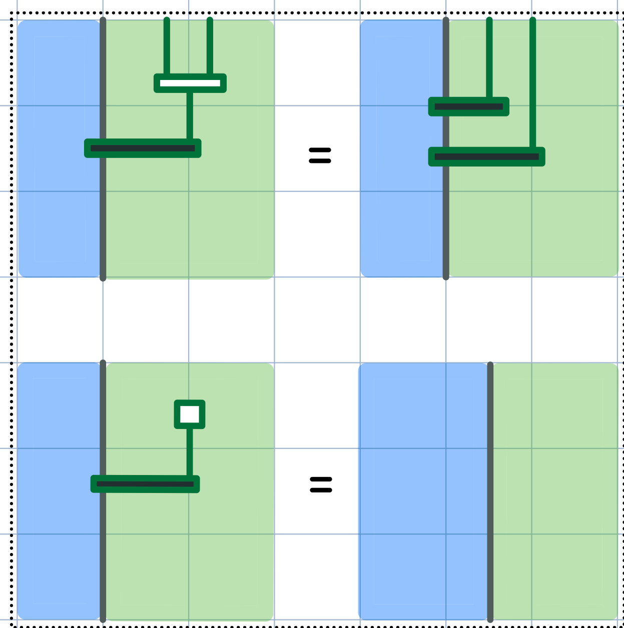
ex: Set

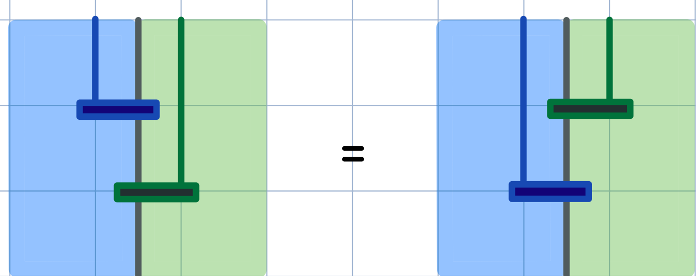






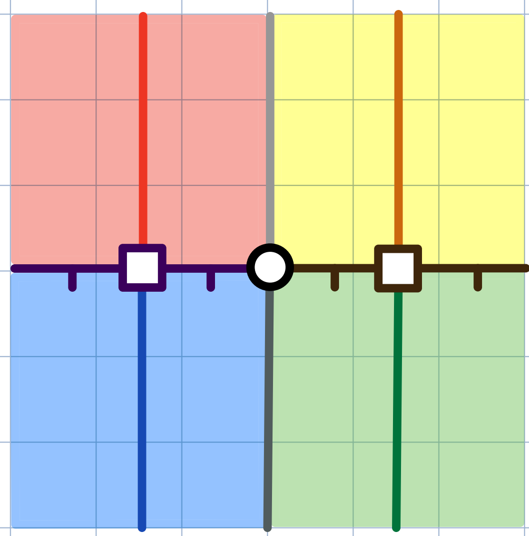




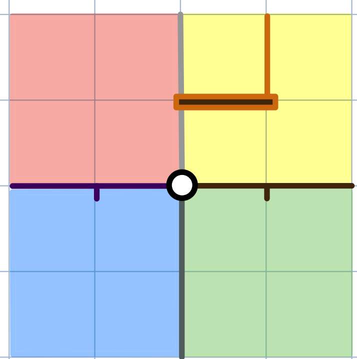


What is an <u>inference</u> between modules?

An inference



with



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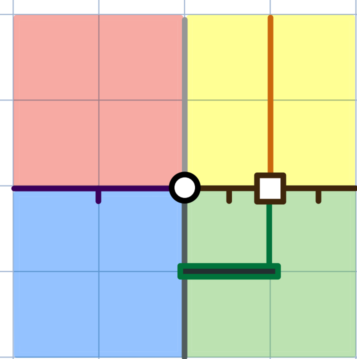
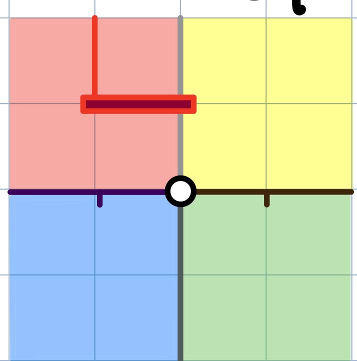


"equivariance"



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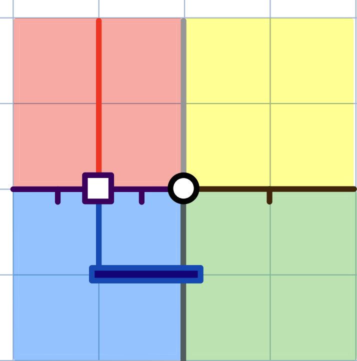
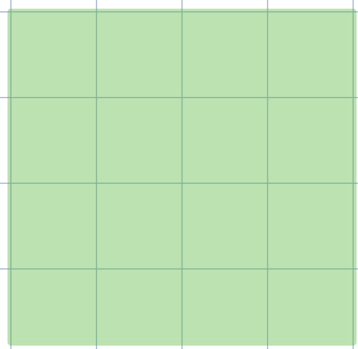


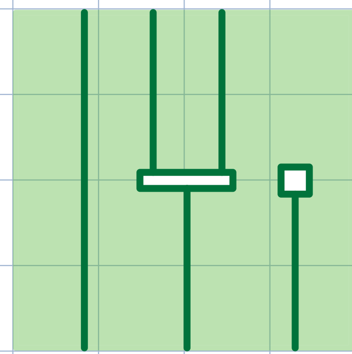
These are the squares of

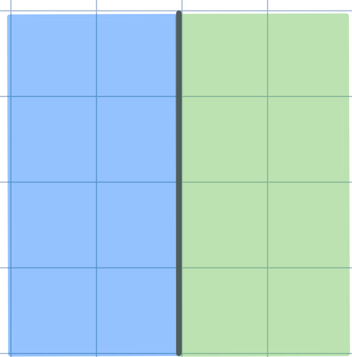
* $\mathrm{Mod}\,\mathbb{C}$. *

Mod $\mathbb{C}$

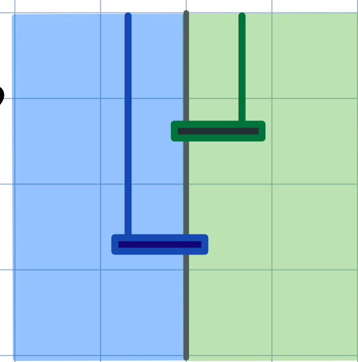


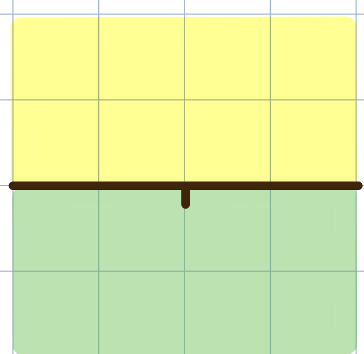
types := monads



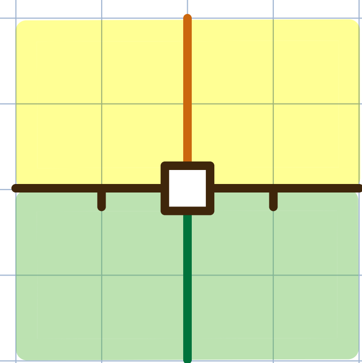


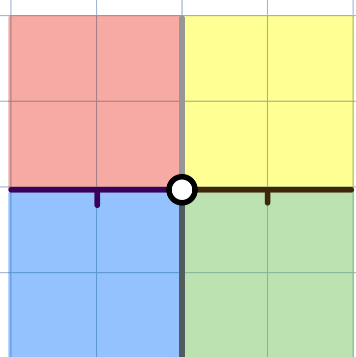
judgements := modules



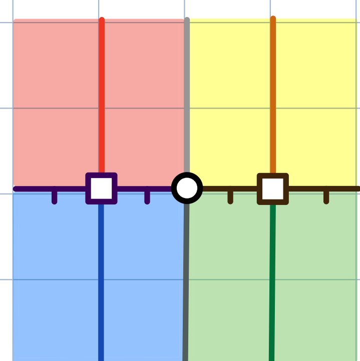


terms := morphisms



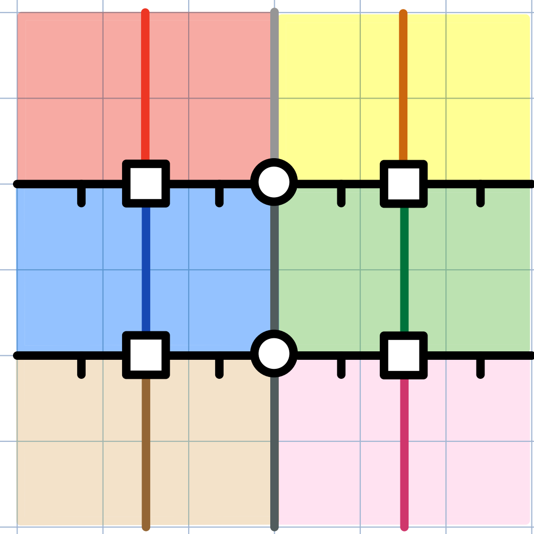


inferences := trans-formations



(note: color)

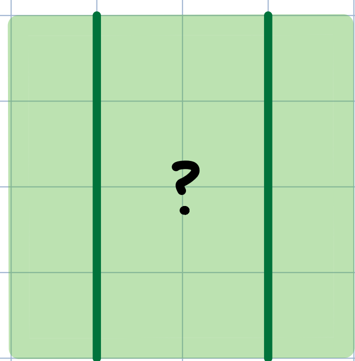
Composition:



sequential is easy.

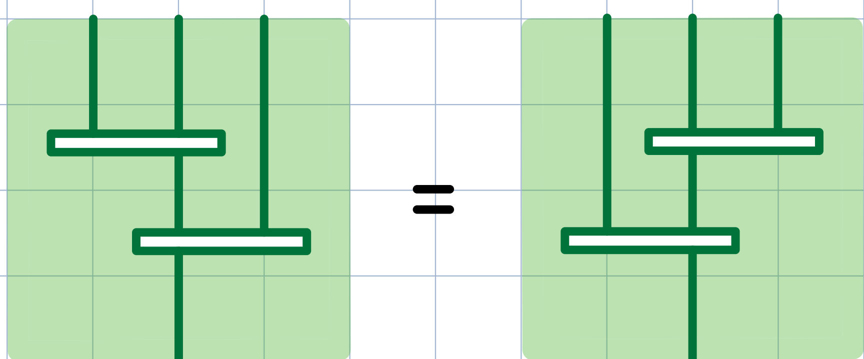
Parallel is more: now, types act.

First, what is the identity?



It must have actions...

...



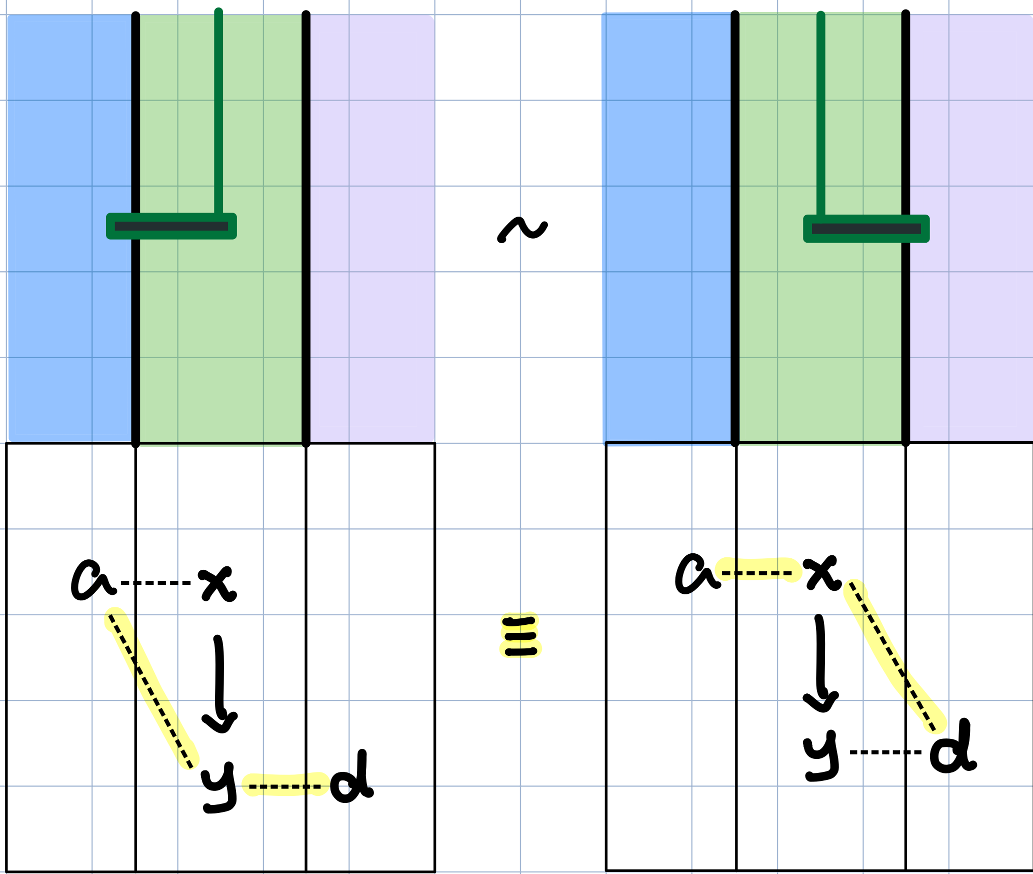
* The monad itself! *

Identity is active & concrete. (r)

Because types act on judgements,

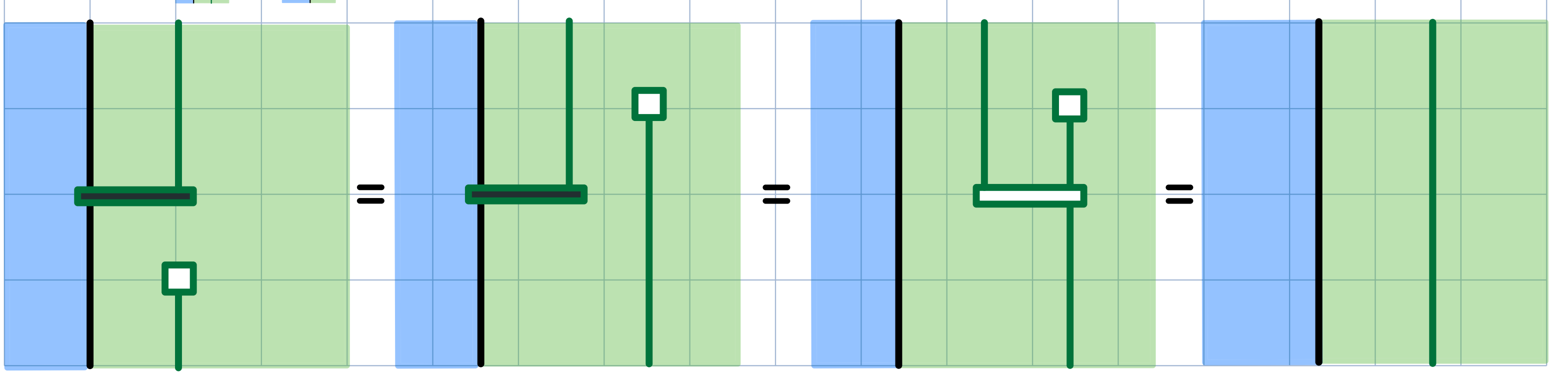
to compose $A \to B \to C$
we must equate the "inner actions" of B.

So, we quotient: (cf. §)



We need this for composition to be unital.

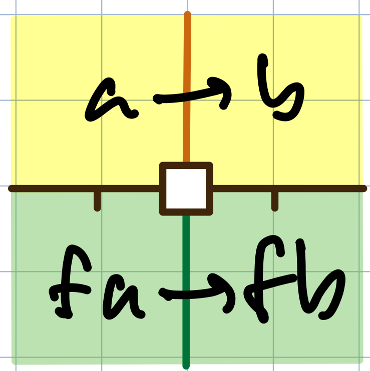
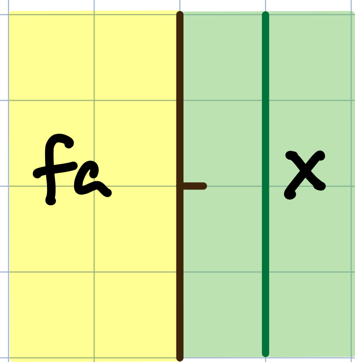
(the iso  ~ )



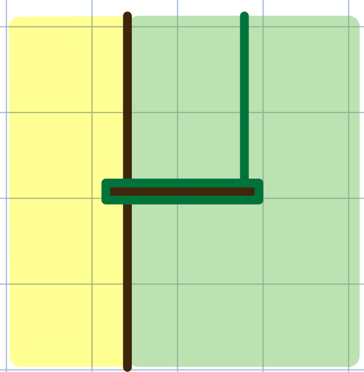
Remember: every function gives an adjoint pair of relations, graph & cograph.

Now, in Mod$\mathbb{C}$: term ⇝ judgement?

$B(f-,-)$

term  ⇝  positive judgement

aka "companion"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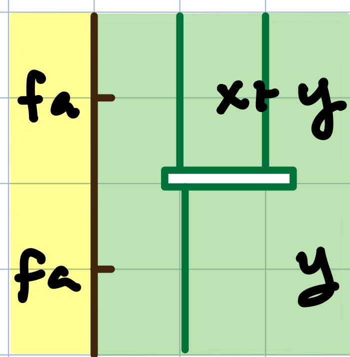

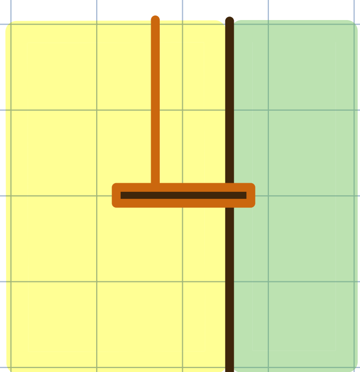
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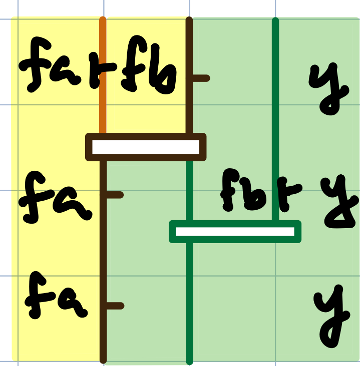
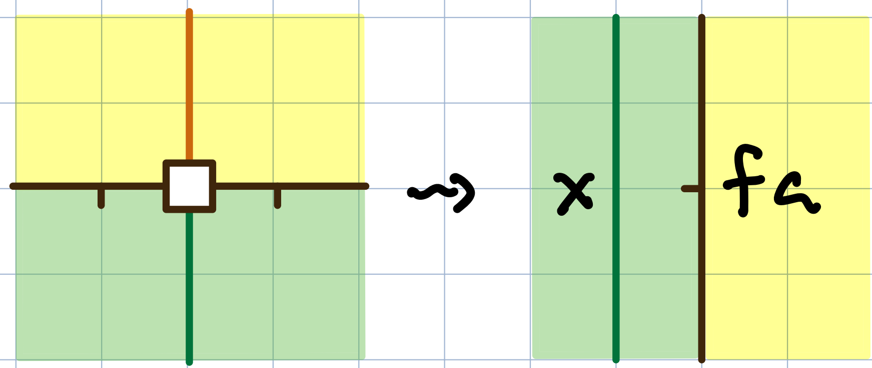


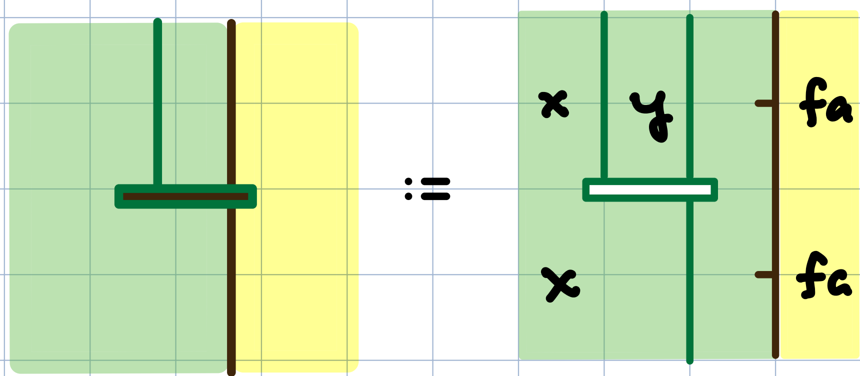
$B[-, f-]$

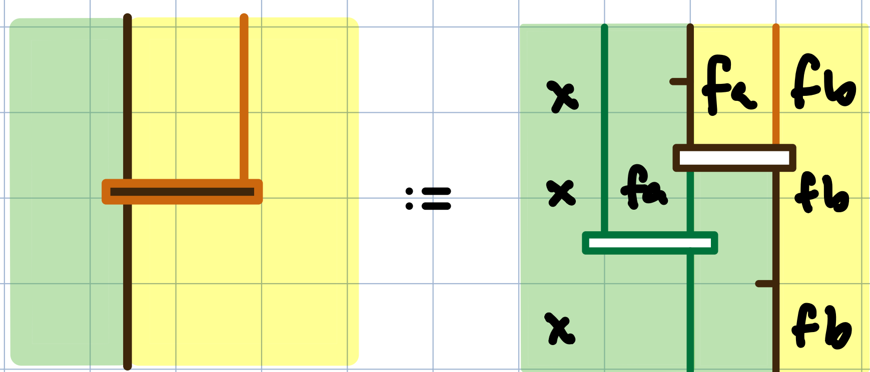
term



negative judgement

aka "conjoint"





puzzle: show that positive is left adjoint to negative.

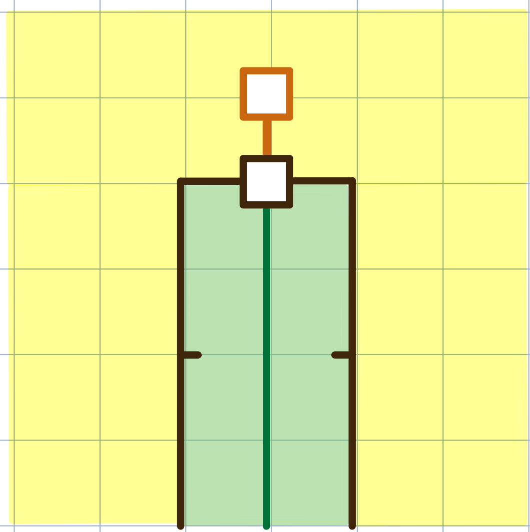
Mod expands logic:

formal $\subset$ actual

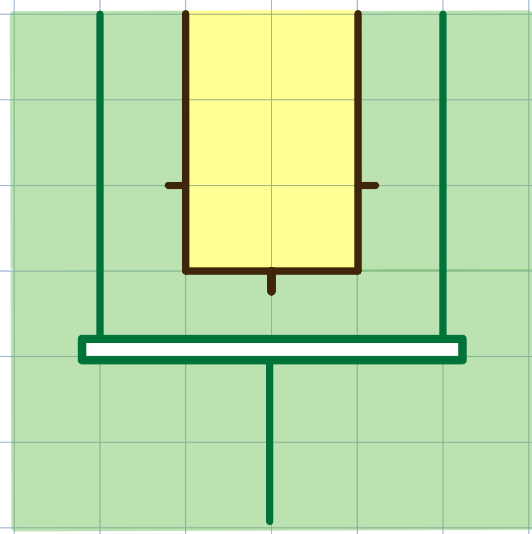
Questions / Thoughts?

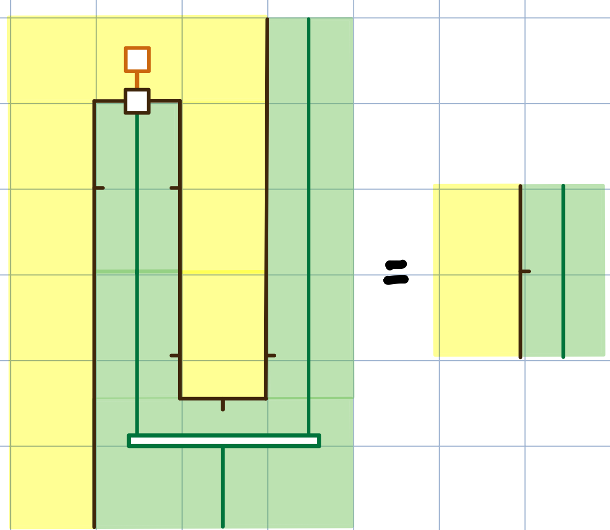
puzzle answer

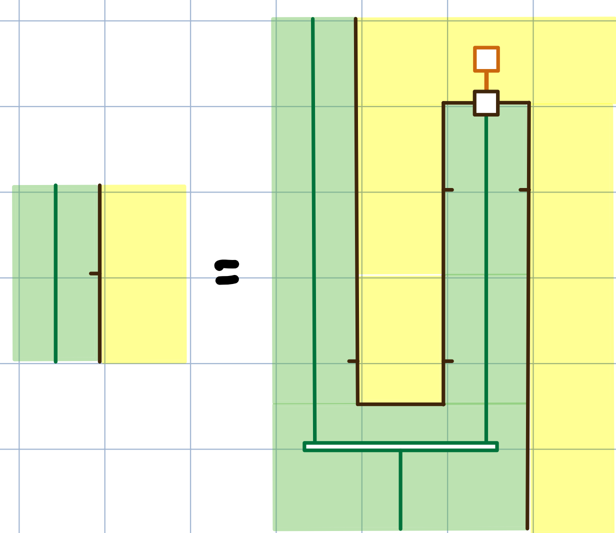
unit



counit

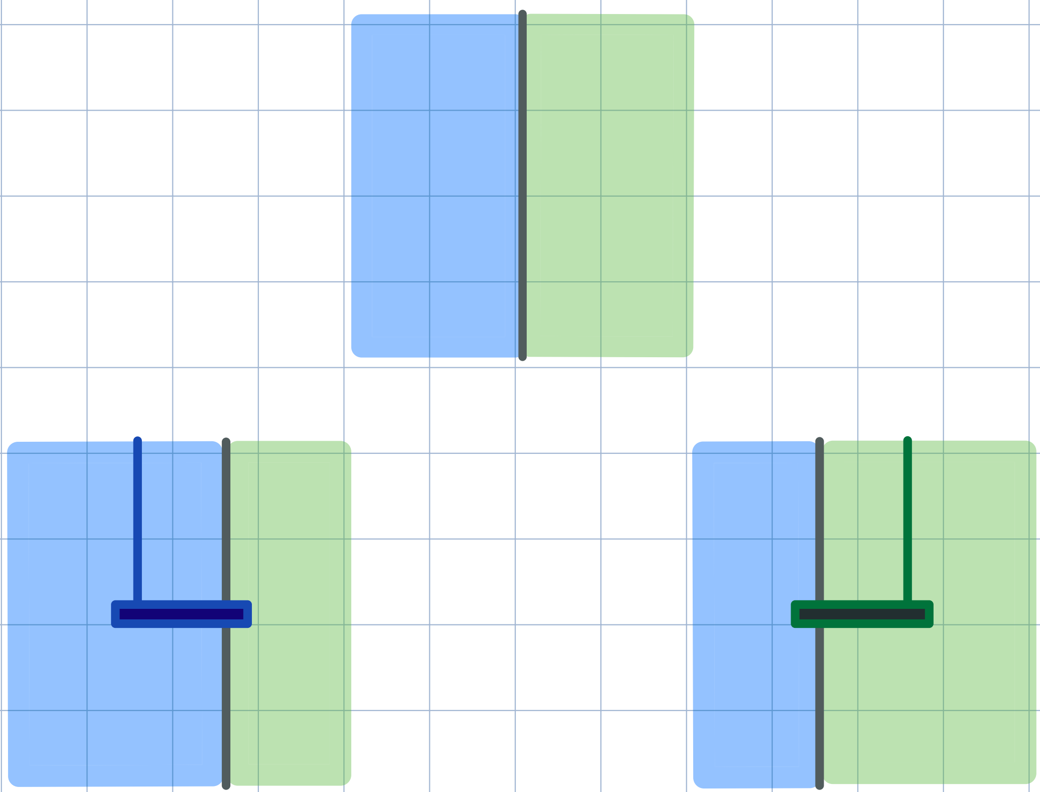






(extra)

if monads in $\mathrm{Mat}\,\mathbb{R}$ are metric spaces, then what is a bimodule?



(equations automatic)